

# Seshadri constants on surfaces of general type

Thomas Bauer and Tomasz Szemberg

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## Abstract

We study Seshadri constants of the canonical bundle on minimal surfaces of general type. First, we prove that if the Seshadri constant  $\varepsilon(K_X, x)$  is between 0 and 1, then it is of the form  $(m-1)/m$  for some integer  $m \geq 2$ . Secondly, we study values of  $\varepsilon(K_X, x)$  for a very general point  $x$  and show that small values of the Seshadri constant are accounted for by the geometry of  $X$ .

## Introduction

Given a smooth projective variety  $X$  and a nef line bundle  $L$  on  $X$ , Demailly defines the Seshadri constant of  $L$  at a point  $x \in X$  as the real number

$$\varepsilon(L, x) =_{\text{def}} \inf_C \frac{L \cdot C}{\text{mult}_x C},$$

where the infimum is taken over all irreducible curves passing through  $x$  (see [3] and [5, Chapt. 5]). If  $L$  is ample, then  $\varepsilon(L, x) > 0$  for all points  $x \in X$ . When  $X$  is a surface, then by a result of Ein and Lazarsfeld [4] one has for ample  $L$

$$\varepsilon(L, x) \geq 1 \quad \text{for all except perhaps countably many points } x \in X. \quad (*)$$

In the present paper we study the Seshadri constants of the canonical bundle  $K_X$  on minimal surfaces of general type (i.e., we study  $\varepsilon(K_X, x)$  on surfaces  $X$  of Kodaira dimension 2 that do not contain any  $(-1)$ -curves, or, equivalently, on surfaces  $X$  whose  $K_X$  is big and nef). Motivated by the result  $(*)$  of Ein-Lazarsfeld we first ask for the potential values below 1 that  $\varepsilon(K_X, x)$  might have. We show:

**Theorem 1** *Let  $X$  be a smooth projective surface such that the canonical divisor  $K_X$  is big and nef. Let  $x$  be any point on  $X$ .*

- (a) *One has  $\varepsilon(K_X, x) = 0$  if and only if  $x$  lies on one of the finitely many  $(-2)$ -curves on  $X$ .*
- (b) *If  $0 < \varepsilon(K_X, x) < 1$ , then there is an integer  $m \geq 2$  such that*

$$\varepsilon(K_X, x) = \frac{m-1}{m},$$

*and there is an irreducible curve  $C \subset X$  such that  $\text{mult}_x(C) = m$  and  $K_X \cdot C = m-1$ . (In other words, the curve  $C$  computes the Seshadri constant of  $K_X$  at  $x$ .)*

- (c) If  $0 < \varepsilon(K_X, x) < 1$  and  $K_X^2 \geq 2$ , then either
- (i)  $\varepsilon(K_X, x) = \frac{1}{2}$  and  $x$  is the double point of an irreducible curve  $C$  with arithmetic genus  $p_a(C) = 1$  and  $K_X \cdot C = 1$ , or
  - (ii)  $\varepsilon(K_X, x) = \frac{2}{3}$  and  $x$  is a triple point of an irreducible curve  $C$  with arithmetic genus  $p_a(C) = 3$  and  $K_X \cdot C = 2$ .
- (d) If  $0 < \varepsilon(K_X, x) < 1$  and  $K_X^2 \geq 3$ , then only case (c)(i) is possible.

It is well known that the bicanonical system  $|2K_X|$  is base point free on almost all surfaces of general type. For such surfaces one gets easily the lower bound  $\varepsilon(K_X, x) \geq 1/2$  for all  $x$  away of the contracted locus. In general however one knows only that  $|4K_X|$  is base point free, which gives a lower bound of  $1/4$ . Theorem 1 shows in particular that one has  $\varepsilon(K_X, x) \geq 1/2$  in all cases. Moreover this bound turns out to be sharp, i.e., there are examples of surfaces  $X$  and points  $x$  such that  $\varepsilon(K_X, x) = 1/2$  (see Example 1.2). We do not know whether all values  $(m-1)/m$  for arbitrary  $m \geq 2$  actually occur. As part (c) of Theorem 1 shows, however, values  $(m-1)/m$  with  $m \geq 4$  can occur only in the case  $K_X^2 = 1$ . We will show in Example 1.3 that curves as in (c)(i) actually exist on surfaces with arbitrarily large degree of the canonical bundle. In other words, one cannot strengthen the result by imposing higher bounds on  $K_X^2$ . It would be interesting to know whether curves as in (c)(ii) exist.

We consider next Seshadri constants of the canonical bundle at a very general point. It is known that the function  $x \mapsto \varepsilon(L, x)$  is lower semi-continuous in the topology on  $X$ , whose closed sets are the countable unions of subvarieties (see [7]). In particular, it assumes its maximal value for  $x$  very general, i.e., away of an at most countable union of proper Zariski closed subsets of  $X$ . This maximal value will be denoted by  $\varepsilon(L, 1)$ . We show:

**Theorem 2** *Let  $X$  be a smooth projective surface such that  $K_X$  is big and nef. If  $K_X^2 \geq 2$ , then*

$$\varepsilon(K_X, 1) > 1.$$

Note that it may well happen that  $\varepsilon(K_X, x) = 1$  for infinitely many  $x \in X$  even when  $K_X^2 \geq 2$ : Consider for instance a smooth quintic surface  $X \subset \mathbb{P}^3$  containing a line  $\ell$ . Then  $\varepsilon(K_X, x) = 1$  for all  $x \in \ell$ . However, one has  $\varepsilon(K_X, 1) > 1$  by [1, Theorem 2.1(a)]. Theorem 2 states that the same conclusion holds in general as soon as  $K_X^2 \geq 2$ .

If  $K_X^2$  is even larger, then further geometric statements are possible. We show:

**Theorem 3** *Let  $X$  be a smooth projective surface such that  $K_X$  is big and nef. If  $K_X^2 \geq 6$ , then*

- (a)  $\varepsilon(K_X, 1) \geq 2$ ,
- (b)  $\varepsilon(K_X, 1) = 2$  if and only if  $X$  admits a genus 2 fibration  $X \rightarrow B$  over a smooth curve  $B$ .

A somewhat more general statement is given in Propositions 2.4 and 2.5.

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## 1 Seshadri constants at arbitrary points

In this section we prove Theorem 1. We will need the following fact:

**Proposition 1.1** *Let  $L$  be a nef and big line bundle on a smooth projective surface, and let  $x$  be a point such that  $\varepsilon(L, x) < \sqrt{L^2}$ . Then there is an irreducible curve  $C$  such that*

$$\varepsilon(L, x) = \frac{L \cdot C}{\text{mult}_x(C)}.$$

The point here is that  $\varepsilon(L, x)$  is in fact computed by a curve rather than being approximated by a sequence of curves.

*Proof.* We fix a real number  $\xi$  such that  $1 < \xi < \frac{\sqrt{L^2}}{\varepsilon(L, x)}$ . By definition of  $\varepsilon(L, x)$  there is in any event a sequence  $(C_n)_{n \in \mathbb{N}}$  of irreducible curves such that  $L \cdot C_n / \text{mult}_x(C_n)$  converges to  $\varepsilon(L, x)$  from above. In particular  $L \cdot C_n / \text{mult}_x(C_n) < \frac{1}{\xi} \cdot \sqrt{L^2}$  for  $n \gg 0$ .

On the other hand the asymptotic Riemann-Roch theorem implies that for  $k \gg 0$  there are divisors  $D \in |kL|$  such that the quotient  $L \cdot D / \text{mult}_x D$  is arbitrarily close to  $\sqrt{L^2}$ , in particular less than  $\xi \cdot \sqrt{L^2}$ . Fixing such a value  $k$ , [1, Lemma 5.2] implies that for all sufficiently large  $n$ , the curve  $C_n$  is a component of  $D$ . Therefore in the sequence  $(C_n)$  there are only finitely many distinct curves, and this implies the assertion.  $\square$

*Proof of Theorem 1.* (a) Suppose first that  $C$  is a rational  $(-2)$ -curve on  $X$ . Then  $K_X \cdot C = 0$  by the adjunction formula, so that clearly

$$\varepsilon(K_X, x) = 0$$

for any point  $x \in C$ . Conversely, suppose that  $\varepsilon(K_X, x) = 0$  for some point  $x \in X$ . Thanks to Proposition 1.1 we have  $K_X \cdot C = 0$  for some irreducible curve  $C$  on  $X$ . As  $K_X^2 > 0$ , we get  $C^2 < 0$  from the index theorem. The adjunction formula then tells us that  $C$  is a  $(-2)$ -curve.

(b) Since by assumption  $0 < \varepsilon(K_X, x) < 1$ , there is in any event an integer  $m \geq 2$  such that

$$\frac{m-2}{m-1} < \varepsilon(K_X, x) \leq \frac{m-1}{m}. \quad (1)$$

We will show that then necessarily  $\varepsilon(K_X, x) = (m-1)/m$ . To this end we start by making use of a result from [1] (cf. Proof of [1, Theorem 3.1]):

If  $L$  is a nef and big line bundle, and if  $\sigma$  is a real number such that  $\sigma L - K_X$  is nef, then for any irreducible curve  $C$  the multiplicity  $\ell = \text{mult}_x(C)$  at a given point  $x \in X$  is bounded as

$$\ell \leq \frac{1}{2} + \sqrt{\frac{(L \cdot C)^2}{L^2} + \sigma L \cdot C + \frac{9}{4}} . \quad (2)$$

(Note that in [1] the line bundle  $L$  is assumed to be ample. The argument proving (2) still holds, however, when  $L$  is merely nef and big.) In our situation we take  $L = K_X$  and  $\sigma = 1$ , and we have by Proposition 1.1 an irreducible curve  $C$  such that

$$\varepsilon(K_X, x) = \frac{K_X \cdot C}{\ell} .$$

Note that since  $\varepsilon(K_X, x)$  is less than 1 we have  $\ell \geq 2$ . Moreover if  $\ell = 2$ , then the only possible value of the Seshadri constant is  $1/2$ . From now on we assume therefore that  $\ell \geq 3$ .

Writing  $d = K_X \cdot C$ , the inequality (2) implies then

$$\frac{K_X \cdot C}{\ell} \geq \frac{d}{\frac{1}{2} + \sqrt{\frac{d^2}{K_X^2} + d + \frac{9}{4}}} . \quad (3)$$

The expression on the right hand side of (3) is increasing both as a function of  $d$  and as a function of  $K_X^2$ . Now, a calculation shows that if  $d \geq m$ , then the right hand side of (3) is bigger than  $(m-1)/m$ . So our assumption (1) implies that

$$d \leq m - 1 .$$

We will now show that in fact  $d = m - 1$ . Indeed, the Seshadri constant  $\varepsilon(K_X, x)$  is the fraction

$$\varepsilon(K_X, x) = \frac{d}{\ell} ,$$

but among all fractions that are smaller than 1 and that have numerator  $d$ , the fraction  $d/(d+1)$  is the biggest. So

$$\varepsilon(K_X, x) \leq \frac{d}{d+1} .$$

But if  $d \leq m - 2$ , then this implies

$$\varepsilon(K_X, x) \leq \frac{m-2}{m-1} ,$$

giving a contradiction with (1). We have thus established the equality

$$d = m - 1 .$$

From the index theorem we obtain

$$K_X^2 C^2 \leq (K_X \cdot C)^2 = d^2 . \quad (4)$$

Note that this implies that in any event  $C^2 \leq d^2$ . The adjunction formula then gives

$$p_a(C) = 1 + \frac{1}{2} C \cdot (C + K_X) \leq 1 + \frac{d(d+1)}{2}.$$

As  $C$  has a point of multiplicity  $\ell$ , the arithmetic genus of its normalization drops by at least  $\binom{\ell}{2}$ . Since  $\ell \geq 3$ , we get therefore

$$\ell \leq d+1 = m.$$

So we conclude that

$$\varepsilon(K_X, x) = \frac{d}{\ell} \geq \frac{d}{m} = \frac{m-1}{m},$$

and this completes the proof of (b).

(c) The argument is parallel to that of part (b) up to the inequality (4). We keep the notations from there. Suppose that  $0 < \varepsilon(K_X, x) < 1$  and  $K_X^2 \geq 2$ . We see from (4) that  $C^2 \leq \frac{1}{2}d^2$ . If we have  $\ell \geq 4$ , then we obtain

$$\ell < d+1 = m, \tag{5}$$

and therefore

$$\varepsilon(K_X, x) = \frac{d}{\ell} > \frac{d}{m} = \frac{m-1}{m},$$

in contradiction with (1).

So it remains to study the cases  $\ell = 2$  and  $\ell = 3$ . If  $\ell = 2$ , then necessarily  $d = K_X \cdot C = 1$ , so that  $\varepsilon(K_X, x) = 1/2$ . The assumption  $K_X^2 \geq 2$  together with (4) imply that  $C^2 < 0$ . The adjunction formula and the fact that  $C$  has a double point imply then that  $p_a(C) = 1$ .

If  $\ell = 3$ , then we arrive at the situation  $d = K_X \cdot C = 2$ ,  $C^2 = 2$ , and hence  $p_a(C) = 3$ .

(d) Keeping notations from the preceding case, the assumption  $K_X^2 \geq 3$  implies now via (4) the inequality  $C^2 \leq \frac{1}{3}d^2$ , so that we arrive for  $\ell \geq 3$  again at (5), which gives a contradiction as before.  $\square$

We now describe an example of a surface  $X$  of general type, where one has

$$K_X^2 = 1 \quad \text{and} \quad \varepsilon(K_X, x) = \frac{1}{2} \quad \text{for some } x \in X.$$

**Example 1.2** Consider a general surface  $X$  of degree 10 in weighted projective space  $\mathbb{P}(1, 1, 2, 5)$ . Then  $X$  is smooth,  $K_X = \mathcal{O}_X(10 - 1 - 1 - 2 - 5) = \mathcal{O}_X(1)$  by adjunction. In particular,  $K_X$  is ample,  $K_X^2 = 1$ , and  $h^0(K_X) = 2$ , which corresponds to the first two variables of weight 1 (see [8, p. 311] for details). The canonical pencil consists of curves of arithmetic genus 2 and has exactly one base point in which all canonical curves are smooth. Blowing up this point  $\sigma : Y \rightarrow X$  we get a genus 2 fibration  $f : Y \rightarrow \mathbb{P}^1$  over the exceptional curve. The mapping  $f$  is a relatively minimal semistable family of curves. In fact there are neither multiple nor reducible fibers possible, as  $K_X^2 = 1$  and  $K_X$  is ample. So the singular fibers, if any, are irreducible curves with double points. If all fibers were smooth genus 2

curves, then the topological Euler characteristic of  $Y$  would be  $c_2(Y) = 2 \cdot (-2) = -4$ . Since  $\chi(Y) = \chi(X) = 3$ , this would contradict the Noether formula. This shows that there exists a singular canonical curve  $D \in |K_X|$ . In the singular point of  $D$  one has the Seshadri quotient  $\frac{1}{2}$ .

We now give an example showing that curves as in part (c)(i) and (d) of Theorem 1 occur on surfaces with arbitrarily large degree of the canonical bundle.

**Example 1.3** In the product  $\mathbb{P}^2 \times \mathbb{P}^2$  we consider a nontrivial conic bundle  $\mathcal{C}_0 \rightarrow \mathbb{P}^2$  over the first factor with a smooth discriminant curve  $\Delta \subset \mathbb{P}^2$ . This can be obtained explicitly as a general divisor of bidegree  $(1, 2)$ . Let  $B \subset \mathbb{P}^2$  be a smooth plane curve of degree  $d \geq 4$  intersecting  $\Delta$  transversally. We restrict  $\mathcal{C}_0$  to a conic bundle  $f : \mathcal{C} \rightarrow B$  over  $B$ . We fix a point  $b_0 \in B \cap \Delta$ . The fiber of  $f$  over  $b_0$  consists of two lines  $L_0$  and  $L_1$  meeting in a point  $P$ . It is easy to find a smooth cubic  $D$  in the plane spanned by  $L_0$  and  $L_1$  not passing through  $P$  and such that it intersects  $L_0$  in three distinct points and such that it cuts out on  $L_1$  a divisor of the form  $2Q + R$ . We extend  $D$  to a hypersurface in  $\mathbb{P}^2 \times \mathbb{P}^2$  simply taking the product  $\mathcal{D} = \mathbb{P}^2 \times D$ . If  $D$  is sufficiently general, then the intersection curve  $\Gamma = \mathcal{C} \cap \mathcal{D}$  is smooth. Taking the double covering  $\sigma : X \rightarrow \mathcal{C}$  branched over  $\Gamma$  we obtain a smooth surface  $X$  with a genus 2 fibration  $\alpha : X \rightarrow B$ . By the subadditivity of Kodaira dimensions  $X$  is of general type. This construction may be considered as reversing the procedure described before Theorem 4.13 in [2]. The fiber over  $b_0$  consists of two reduced and irreducible curves  $C_0$  and  $C_1$  lying over  $L_0$  and  $L_1$  respectively. The curve  $C_0$  is a smooth elliptic curve, while  $C_1$  has a double point over  $Q$  and arithmetic genus 1. Since  $(C_0 + C_1)^2 = 0$  and  $(C_0 + C_1) \cdot K_X = 2$ , we see by adjunction that  $C_1^2 = -1$  and  $K_X \cdot C_1 = 1$ . Hence  $C_1$  is a curve as in part (c) of Theorem 1. The formula in [2, Theorem 4.13] shows that  $K_X^2$  grows linearly with the genus of  $B$ , so that taking the curve  $B$  of high enough degree we can make  $K_X^2$  arbitrarily large.

## 2 Seshadri constants at very general points

In this section we prove Theorem 2, which is stated in the introduction. We start by recalling the following result obtained by Syzdek and the second author [9]. The result is stated in [9] for ample line bundles, but the proof works verbatim for big and nef ones.

**Theorem 2.1** *Let  $L$  be a big and nef line bundle on a smooth projective surface  $X$ . Assume that*

$$\varepsilon(L, 1) < \sqrt{\frac{7}{9}L^2}.$$

*Then  $X$  is either a smooth cubic in  $\mathbb{P}^3$  or  $X$  is fibred by curves computing  $\varepsilon(L, x)$ .*

*Proof of Theorem 2.* We claim first that there are at most finitely many reduced and irreducible curves  $C \subset X$  with  $K_X \cdot C \leq 1$ . To see this, consider first the case that  $K_X \cdot C = 0$ . Then  $C$  is a  $(-2)$ -curve, and we know that there are only finitely many of them on  $X$ . Next, suppose  $K_X \cdot C = 1$ . Then the index theorem gives  $K_X^2 C^2 \leq (K_X \cdot C)^2 = 1$ , which implies  $C^2 \leq 0$ . From the genus formula

$p_a(C) = 1 + \frac{1}{2}(C^2 + K_X \cdot C)$  we see that  $C^2$  is an odd number. So we have  $C^2 \leq -1$ , and therefore  $C$  is the only irreducible curve in its numerical equivalence class. The claim now follows from the fact that there are only finitely many classes in the Neron-Severi group of  $X$  that have degree 1 with respect to  $K_X$ .

Now assume to the contrary that  $\varepsilon(K_X, 1) = 1$ . Then the numerical assumptions of Theorem 2.1 are satisfied for the line bundle  $L = K_X$ . Since a cubic in  $\mathbb{P}^3$  is not of general type, there must be a fibration of curves computing  $\varepsilon(K_X, x)$ . Since any curve  $C$  in the fibration is smooth in its general point it must be  $K_X \cdot C = 1$  which contradicts above reasoning on the number of such curves.  $\square$

The proof shows that in the situation of the theorem one has in fact the lower bound  $\varepsilon(K_X, x) \geq \frac{1}{3}\sqrt{14}$ . It is unlikely, however, that this particular bound is sharp.

Theorem 2 gives in particular the following interesting characterization of the situations in which equality holds in statement (\*) at the beginning of the introduction.

**Corollary 2.2** *Let  $X$  a smooth projective surface such that  $K_X$  is big and nef. Then  $K_X^2 = 1$  if and only if  $\varepsilon(K_X, 1) = 1$ .*

**Remarks 2.3** (i) The corollary should be seen in light of the fact that in general it is very well possible to have ample line bundles  $L$  on smooth projective surfaces such that  $\varepsilon(L, x) = 1$  for very general  $x$ , while  $L^2$  can be arbitrarily large. Consider for instance a product  $X = C \times D$  of two smooth irreducible curves, and denote by a slight abuse of notation the fibers of both projections again by  $D$  and  $C$ . The line bundles  $L_m = mC + D$  are ample and we have  $L_m \cdot C = 1$ , so that in any event  $\varepsilon(L_m, x) \leq 1$  for every point  $x \in X$ . One has in fact  $\varepsilon(L_m, x) = 1$ , which can be seen as follows: If  $F$  is any irreducible curve different from the fibers of the projections with  $x \in F$ , then we may take a fiber  $D'$  of the first projection with  $x \in D'$ , and we have

$$L_m \cdot F \geq D' \cdot F \geq \text{mult}_x(D') \cdot \text{mult}_x(F) \geq \text{mult}_x(F)$$

which implies  $\varepsilon(L_m, x) \geq 1$ . So  $\varepsilon(L_m, x) = 1$ , but on the other hand  $L_m^2 = 2m$  is unbounded.

(ii) Consider for a moment a minimal surface  $X$  of general type such that  $p_g = 0$ . One knows then that the bicanonical system  $|2K_X|$  is composed with a pencil if and only if  $K_X^2 = 1$  (see [6, Theorem 3.1]). The corollary shows that this geometric condition is also encoded in Seshadri constants through the condition  $\varepsilon(K_X, 1) = 1$ .

In the spirit of Theorem 2 we now show the following result.

**Proposition 2.4** *Let  $X$  be a smooth projective surface such that  $K_X$  is big and nef. If  $K_X^2 \geq 6$ , then either*

- (a)  $\varepsilon(K_X, 1) > 2$ , or
- (b)  $\varepsilon(K_X, 1) = 2$ , and there exists a pencil of curves of genus 2 computing  $\varepsilon(K_X, x)$  for  $x$  very general.

*Proof.* If  $\varepsilon(K_X, 1) > 2$ , then we are done. So let us assume that  $\varepsilon(K_X, 1) \leq 2$ . Then the numerical assumptions of Theorem 2.1 are satisfied, and hence there is a fibration of curves computing  $\varepsilon(K_X, 1)$ . Since a curve in the fibration is smooth in its general point  $x$ ,  $\varepsilon(K_X, x)$  is an integer. By Theorem 2 we have then  $\varepsilon(K_X, 1) = 2$ , and  $\varepsilon(K_X, x)$  is computed by a curve  $C$  satisfying  $K_X \cdot C = 2$  and  $C^2 = 0$ . Hence  $C$  is a member of a pencil of curves of genus 2.  $\square$

The presence of a genus 2 fibration on a surface of general type is the typical obstacle in the positivity of the canonical bundle and the associated maps. Hence it is not very surprising that it implies an upper bound on  $\varepsilon(K_X, 1)$ , which can be regarded as a converse of Proposition 2.4(b).

**Proposition 2.5** *Let  $X$  be a smooth minimal surface of general type such that there is a genus 2 fibration  $f : X \rightarrow B$  over a smooth curve  $B$ . Then*

$$\varepsilon(K_X, 1) \leq 2 ,$$

*and if  $K_X^2 \geq 4$ , then actually*

$$\varepsilon(K_X, 1) = 2 .$$

*Proof.* If  $K_X^2 \leq 4$ , then the assertion is clear, because in any event one has  $\varepsilon(K_X, x) \leq \sqrt{K_X^2}$  for all  $x \in X$  by Kleiman's theorem (cf. [5, Proposition 5.1.9]). So we may assume that  $K_X^2 \geq 5$ . Let  $F$  be a generic fiber of  $f$ . As  $K_X$  restricts to the canonical bundle on  $F$ , we have  $K_X \cdot F = 2$ . In order to conclude we need to show that for a general point  $x$  the quotient  $\frac{K_X \cdot C}{\text{mult}_x C}$  is greater or equal 2 for all irreducible curves passing through the point  $x$ . Since on a surface of general type there is no elliptic or rational curve passing through a general point, we have by adjunction  $K_X \cdot C \geq 2$ . Suppose for a contradiction that  $\frac{K_X \cdot C}{\text{mult}_x C} < 2$  holds. We must then in particular have  $\text{mult}_x C \geq 2$ . As  $\frac{K_X \cdot C}{\text{mult}_x C} < \sqrt{K_X^2}$ , the argument from the proof of the main theorem in [4] shows that there exists a nontrivial family of pointed curves  $(C, x)$  with multiplicity  $m \geq 2$  at  $x$  such that  $\frac{K_X \cdot C}{m} < 2$ . Then  $K_X \cdot C \leq 2m - 1$  and, by [4, Corollary 1.2],  $C^2 \geq m(m - 1)$ . Combining this with the index theorem we have

$$5m(m - 1) \leq K_X^2 C^2 \leq (K_X \cdot C)^2 \leq (2m - 1)^2 ,$$

which is impossible for  $m \geq 2$ .  $\square$

Propositions 2.4 und 2.5 imply Theorem 3 from the introduction.

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Thomas Bauer, Fachbereich Mathematik und Informatik, Philipps-Universität Marburg,  
Hans-Meerwein-Straße, D-35032 Marburg, Germany.  
E-mail: [tbauer@mathematik.uni-marburg.de](mailto:tbauer@mathematik.uni-marburg.de)

Tomasz Szemberg, Instytut Matematyki AP, PL-30-084 Kraków, Poland  
E-mail: [szemberg@ap.krakow.pl](mailto:szemberg@ap.krakow.pl)